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NONLINEAR OSCILLATOR UNDER EXTERNAL ASYNCHRONOUS INFLUENCE: COMPARISON OF CANONICAL AND NON-CANONICAL PERTURBATION METHODS OF ANALYSIS [+)]

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Abstract

A non-canonical (non-Hamiltonian) perturbation method for study of nonlinear oscillator under external asynchronous action in variables "energy-angle" is presented. As new variables, the iteration constants of the original solution are introduced. Consistently applying the method of canonical transformations and producing functions, a canonical approach in "action-angle" variables is developed for analysis of the same system under similar conditions. Both approaches are characterized by the transition, in the very beginning, to functions with constant period and only then the necessary functional matrices are introduced. The same problem is studied by Kuzmak's method, characterized by the opposite approach: first, a functional square matrix is introduced, and only then a transition to functions with a constant period is made. A comparison of the results obtained using the three above-mentioned methods and approaches is made. It is shown that the solutions in the first approximation lead to equal results. In particular, this conclusion is a contribution to the idea that there is no essential difference between non-canonical (non-Hamiltonian) and canonical (Hamiltonian) methods. However, attention is drawn to the fact that the other analytical methods developed in the frame of the Theory of Nonlinear Oscillations could not give, even in the first approximations, a complete coincidence with the solution obtained using the three above-mentioned methods.

The analysis of oscillations and vibrations reduces to the problem of a nonlinear oscillator, subjected to external periodic influence (perturbation). With the development of perturbation methods, two main directions have formed: canonical (Hamiltonian) methods and non-canonical (non-Hamiltonian) methods.

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The methods using mainly canonical transformations in action-angle variable developed earlier. This was in response to the needs of celestial mechanics – see [1]. The method of Lindstedt-Poincare as well as other methods were developed. The second direction of development of the perturbation methods are the methods of the averaged Lagrangian and the averaged Hamiltonian [2]. An overview and a modern presentation of the canonical methods are given in [3-5].

In the first half of the twentieth century, mainly the non-canonical (non-Hamiltonian) perturbation methods were developed for the purpose of analysis of nonlinear electric circuits [6-9]. M.Kruskal has developed a non-canonical theory showing that it can be equivalent to the canonical theory [10]. The development of these methods is reflected in [4, 11-13].

In the application of perturbation and in particular of the asymptotic methods a transformation is made to a generating solution with constant period $P_0 = 2\pi$. A number of methods have been developed for this purpose. In the case of non-canonical methods, the integration constants of the generating solution serve as the new variables while in the case of the canonical methods with the aid of a canonical transformation the treatment is done in action-angle variables (these two approaches are used below). A third possible method is the regularized Euler method introducing a new independent variable [14-16]. After the transformation the generating (non-perturbed) equation coincides with the equation of the harmonic oscillator. K.A.Samoylo has suggested the so-called method of non-linear transformation employing the transformation of the dependent (coordinate) as well as the independent (time) variables. In this case the generating equation again coincides with the equation of the harmonic oscillator. Finally, a method has been developed in which the generating solution has a variable period (dependent on the amplitude). In this case the solution of the variation equation contains secular terms which at a later stage are compensated. This method is by G.E. Kuzmak [18] and has been further developed in a number of works, i.e. [19-23].

We should also mention a number of modifications of the perturbation methods used for analysis of non-linear waves as well as a number of perturbation methods based on conservation laws. In the analysis of solution phenomena perturbation methods are used which based on the Inverse Scattering Method (ISM) for equations such as the Korteweg-de-Vries, the Sine-Gordon, the non-linear Schroedinger equation etc. [26,27].

The present work compares the results obtained through different perturbation methods. It is shown that the solutions using energy-angle variables and action-angle variables (via canonical transformations) and by Kuzmak's method to first approximation lead to equivalent results. This confirms the idea that between the canonical and the non-canonical perturbation methods used in particular for the analysis of an asynchronous oscillator there isn't any principal difference.

1. Generating solution

Let us consider a generalized nonlinear oscillator described by the following system of equations:

$$(1)\qquad \begin{cases} \dfrac{dx}{dt} - p = 0 \\ \dfrac{dp}{dt} + f(x,T) = \mu F_v\left(\dfrac{dx}{dt}, x, t, T\right) \end{cases}$$

where $0 \le \mu << 1$ is a small parameter, T is secondary scaling /slow time/, $T = T_o + \mu t$, $T_o = const$, $dT/dt = \mu$. The secondary scaling (slow time) could represent the slow change of the oscillator parameters: i.e. modulation of the oscillator inductance or capacity, a drift in the supplied power etc. We will take $f(0,T) = 0$. We will seek a solution of equation (1) for x belonging to the interval satisfying $xf(x,T) \ge 0$.

The solution of the system of equations (1) for $\mu = 0$ represents the so called **generating solution** which we will represent as:

$$(2)\qquad X = X_a(E, t + t_o, T), \quad p = p_a(E, t + t_o, T),$$

Here E = const and $t_o = const$ are the constants of integration.

We introduce a circular frequency into the generating solution (2), as follows:

$$\omega(E,T) = \frac{2\pi}{\Pi(E,T)}, \quad \text{where} \qquad \Pi(E,T) = 2\int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2E - V(x,T)}}$$

is the period in time t and $V(x,T) = \int_0^x \sqrt{f(x',T)dx'}$ is the potential energy,

$$V(x_{\min}, T) = V(x_{\max}, T) = E(T)$$

An angle variable Ψ and an integration constant $\alpha = const$ are introduced through the expressions: $t = \dfrac{\Psi}{\omega(E,T)}$, $t_o = \dfrac{\alpha}{\omega(E,T)}$.

Let $\theta = \Psi + \alpha$. We introduce the new functions:

$$(3)\qquad \begin{cases} x = x_b(E,\theta,T) = x_a\left(E, \dfrac{\theta}{\omega(E,T)}, T\right) \\ p = p_b(E,\theta,T) = p_a\left(E, \dfrac{\theta}{\omega(E,T)}, T\right) \end{cases}$$

which are periodic in Ψ and θ with period $p_0 = 2\pi$, independent of E. Therefore the derivatives $\partial x_b/\partial\theta$, $\partial p_b/\partial\theta$, $\partial x_b/\partial E$ and $\partial p_b/\partial E$ are periodic, i.e. they do not contain secular terms.

The system of equations (1) takes the form:

(4)
$$\left|\begin{array}{l} \omega(E,T)\dfrac{\partial x_b(E,\theta,T)}{\partial\theta} - p_b(E,\theta,T) = 0, \\ \omega(E,T)\dfrac{\partial p_b(E,\theta,T)}{\partial\theta} + f(x_b,T) = 0, \end{array}\right.$$

or

(5)
$$\mathbf{Y}\begin{bmatrix} 0 \\ \omega \end{bmatrix} + \begin{bmatrix} -p_b \\ f(x_b,T) \end{bmatrix} = 0,$$

where

(6)
$$\mathbf{Y}(E,\theta,T) = \begin{bmatrix} \dfrac{\partial x_b(E,\theta,T)}{\partial E} & \dfrac{\partial x_b(E,\theta,T)}{\partial\theta} \\ \dfrac{\partial p_b(E,\theta,T)}{\partial E} & \dfrac{\partial p_b(E,\theta,T)}{\partial\theta} \end{bmatrix}.$$

The matrix $\mathbf{Y}$ is periodic as the period $p_o = 2\pi$ is constant and it satisfies the condition for the absence of secular terms.

2. Perturbation in energy-angle variables

We now perturb equation (1) at $\mu \neq 0$. We vary the constant parameters taking $E = E(t)$ and $\alpha = \alpha(t)$.

Taking into account (4) we obtain the following system of equations, equivalent to system (1):

(7)
$$\frac{d\Psi}{dt} = \omega(E,T); \quad \theta(t) = \Psi(t) + \alpha(t)$$

(8)
$$\begin{bmatrix} \dfrac{dE}{dt} \\ \dfrac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \omega(E,T) \end{bmatrix} + \mu\begin{bmatrix} G_r(E,\theta,t,T,\mu) \\ G_s(E,\theta,t,T,\mu) \end{bmatrix}$$

where $\begin{bmatrix} G_r \\ G_s \end{bmatrix} = \mathbf{Y}^{-1}\begin{bmatrix} -\dfrac{\partial x_b}{\partial T} \\ -\dfrac{\partial p_b}{\partial T} + F_v \end{bmatrix}.$

Here, the inverse matrix $\mathbf{Y}^{-1} = \begin{bmatrix} f(x_b,T) & p_b \\ \omega \dfrac{\partial p_b}{\partial E} & -\omega \dfrac{\partial x_b}{\partial E} \end{bmatrix}$ and correspondingly $\det \mathbf{Y} = -1/\omega$, i.e. the condition for the application of the perturbation approach, $\det \mathbf{Y} \neq 0, \infty$ is valid.

We will seek the solution (7) in the form of an asymptotic series:

(9)
$$\begin{cases} E(t) = E_o(t) + \mu E_1(t) + \mu^2 E_2(t) + \dots \\ \theta(t) = \theta_o(t) + \mu\theta_1(t) + \mu^2\theta_2(t) + \dots \end{cases}$$

Substituting (9) into (8), expanding in the powers of μ and equating the coefficients multiplying the same powers of μ^k, we obtain:

(10)
$$\begin{bmatrix} \dfrac{dE_k}{dt} \\ \dfrac{d\theta_k}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \delta_k \end{bmatrix} + \begin{bmatrix} G_{r,k} \\ G_{s,k} \end{bmatrix}, \qquad k = 0, 1, 2, 3, \dots,$$

where δ_k reflects the necessary corrections to ω due to different order of approximation for E in $\omega(E,T)$, i.e.:

$$\mu^k \delta_k = \omega(E_o + \mu E_1 + \dots + \mu^{k-1} E_{k-1} + \mu^k E_k, T) - \omega(E_o + \mu E_1 + \dots + \mu^{k-1} E_{k-1}, T$$
$$G_{r,k} = \hat{G}_{r,k}(E_o, E_1, \dots, E_{k-1}, \theta_o, \theta_1, \dots, \theta_{k-1}, T)$$
$$G_{s,k} = \hat{G}_{s,k}(E_o, E_1, \dots, E_{k-1}, \theta_o, \theta_1, \dots, \theta_{k-1}, T)$$

Here $\hat{G}_{r,k}$ and $\hat{G}_{s,k}$ are differential operators which are applied to the obtained in the previous steps functions.

The complete determination of E_k is possible only when the coefficients of order μ^{k+1} are taken into account. The correction δ_k (10) contains E_k. This is why we solve the equation E_k simultaneously with the equation for θ_{k-1}, i.e. instead of solving (10) we should solve the system of equations:

$$\begin{cases} \dfrac{dE_k}{dt} = G_{r,k}(E_o, E_1, \dots, E_{k-1}, \theta_o, \theta_1, \dots, \theta_{k-1}, t, T) \\ \dfrac{d\theta_{k-1}}{dt} = \delta_{k-1} + G_{r,k-1}(E_o, E_1, \dots, E_{k-2}, \theta_o, \theta_1, \dots, \theta_{k-2}, t, T) \end{cases}$$

In (9) we do the following substitution:

(11) $$\begin{cases} E_k(t) = L_k(t) + U_{rk}(t,T) \\ \theta_k(t) = \alpha_k(t) + U_{sk}(t,T), \\ \dfrac{d\alpha_k(t)}{dt} = \omega_k(T) \end{cases} \qquad k=1,2,3,\ldots$$

where $U_{ro} = 0$, $U_{so} = 0$. This representation takes into account that E is a slow variable while θ is a quick variable. The quantity $L_{k'}(t)$ is obtained only when the coefficients of order μ^{k+1} are considered.

Taking into account equation (11) the perturbation approach reduces equation (10) to:

(12) $$\begin{cases} \dfrac{dL_{k-1}(T)}{dT} + \dfrac{\partial U_{rk}(t,T)}{\partial t} = G_{rk}(t,T) \\ \omega_k(T) + \dfrac{\partial U_{sk}(t,T)}{\partial t} = \delta_k(t,T) + G_{sk}(t,T) \end{cases}$$

We assume that the right-hand sides of (12) are expressed in functions the form of which was found in the previous steps.

From the condition of periodicity of U_{rk} and U_{sk} it follows that:

(13) $$\begin{cases} \dfrac{dL_{k-1}(T)}{dT} = \langle G_{rk} \rangle_t \\ \omega_k(T) = \langle \delta_k + G_{sk} \rangle_t \end{cases}$$

where $< >_t$ means averaging with respect to time.

From (12) $L_{k-1}(T)$ and $\omega_{k'}(T)$ are determined. Then we find U_{rk}, U_{sk} from the following system of equations:

$$\begin{cases} \dfrac{\partial U_{rk}(t,T)}{\partial t} = G_{rk}(t,T) - \langle G_{rk} \rangle_t \\ \dfrac{\partial U_{sk}(t,T)}{\partial t} = \delta_k(t,T) - G_{sk}(t,T) - \langle \delta_k + G_{sk} \rangle_t \end{cases}$$

We should at this point note that instead of t we can use Ψ as an independent variable. In the case the system of equations (7) and (8) is equivalent to:

(14) $$\frac{dt}{d\Psi} = \frac{1}{\omega(E,T)}$$

(15) $$\begin{bmatrix} \dfrac{dE}{d\Psi} \\ \dfrac{d\theta}{d\Psi} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu\left(\frac{1}{\omega}\right)\mathbf{Y}^{-1} \begin{bmatrix} -\dfrac{\partial x_b}{\partial T} \\ -\dfrac{\partial p_b}{\partial T} + F_v \end{bmatrix}$$

The described above perturbation method can be used for the analysis of equations (14) and (15).

3. Perturbation in action-angle variables

Above we found a solution in energy-angle variables (E, θ). The crucial step was the treatment in variables x_b (E, θ, T) and p_b (E, θ, T), having a constant period $p_0 = 2\pi$ with respect to the angle (quick) variable θ. Here instead of energy E we will use the action I. Our goal is to transform canonically in such a way that the new action $I \equiv \overline{P} = const$ to be a constant while the new coordinate $\Psi \equiv \overline{X}$ is linear in time.

For the case in consideration the generating equation can be represented with the canonical equations of Hamilton:

(16)
$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = P; \quad \frac{dP}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} = -f(x,T)$$

For the transition to action-angle variables and the achievement of the set above goal we introduce the generating function $W(x,I,T)$ such that:

$$P = \frac{\partial W(x,I,T)}{\partial t}, \quad \Psi = \frac{\partial W(x,I,T)}{\partial I}$$

$$W(x,I,T) = \pm\int_0^x \sqrt{2E(I,T) - 2V(x',T)}dx'$$

Then the new Hamiltonian is:

$\overline{H}(I,\Psi,T) = \overline{H}(I,T) = H(P,x,T) = E(T)$, where $E(T)$ is the energy integral, $E(T) = p^2/2 + \mathrm{V(x,T)}$.

The period in Ψ must be a constant and equal to 2π. The circular frequency in the generating solution is:

(17)
$$\omega_c(I,T) = \partial\Psi/\partial t = 2\pi/\Pi[E(I,T), T],$$

where the period in time $t = \Psi/\omega_c$ is:

$$\Pi(E,T) = 2\pi\frac{\partial I(E,T)}{\partial E} = 2\frac{\partial}{\partial E}\int_{\min x}^{\max}\sqrt{2E - 2V(x,T)}dx$$

We introduce the new functions:

(18)
$$\begin{cases} x = x_c(I,\theta,T) = x_b[E(I,T),\theta,T] \\ p = p_c(I,\theta,T) = p_b[E(I,T),\theta,T] \end{cases}$$

Taking into account (16)-(18), the system of equations (1) takes the form:

$$\begin{cases} \omega_c(I,T)\left[\dfrac{\partial x_c(I,\theta,T)}{\partial\theta}\right] - p_c(I,\theta,T) = 0 \\ \omega_c(I,T)\left[\dfrac{\partial p_c(I,\theta,T)}{\partial\theta}\right] + f(x_c,T) = 0 \end{cases}$$

or

(19) $$\mathbf{Z}\begin{bmatrix} 0 \\ \omega_c \end{bmatrix} + \begin{bmatrix} -p_c \\ f(x_c,T) \end{bmatrix} = 0$$

where $\mathbf{Z}(I,\theta,T) = \begin{bmatrix} \dfrac{\partial x_c(I,\theta,T)}{\partial I} & \dfrac{\partial x_c(I,\theta,T)}{\partial\theta} \\ \dfrac{\partial p_c(I,\theta,T)}{\partial I} & \dfrac{\partial p_c(I,\theta,T)}{\partial\theta} \end{bmatrix}$.

We seek the solution of the perturbed system of equations (1) by varying the constant parameters $I = I(t)$ and $\alpha = \alpha(t)$ as:

(20) $$\begin{cases} x = x_c[I(t), \Psi(t) + \alpha(t), T] \\ p = p_c[I(t), \Psi(t) + \alpha(t), T] \\ \dfrac{d\Psi}{dt} = \omega_c[I(t), T]; \quad \theta(t) = \Psi(t) + \alpha(t) \end{cases}$$

Substituting (20) in (1) and taking into consideration (19) we obtain:

(21) $$\begin{bmatrix} \dfrac{dI}{dt} \\ \dfrac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_c(I,T) \end{bmatrix} + \mathbf{Z}^{-1}\begin{bmatrix} -\mu\dfrac{\partial x_c(I,\theta,T)}{\partial T} \\ -\mu\dfrac{\partial p_c(I,\theta,T)}{\partial T} + \mu F_\nu \end{bmatrix}$$

where $\mathbf{Z}^{-1} = \begin{bmatrix} \dfrac{1}{\omega_c} f(x_c,T) & \dfrac{1}{\omega_c} p_c \\ \dfrac{\partial p_c}{\partial I} & -\dfrac{\partial x_c}{\partial I} \end{bmatrix}$, $\det \mathbf{Z} = -1$.

The system (21) can be expressed as:

(22) $$\begin{bmatrix} \dfrac{dI}{dt} \\ \dfrac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \delta \end{bmatrix} + \begin{bmatrix} G_v(I,\theta,t,T,\mu) \\ G_w(I,\theta,t,T,\mu) \end{bmatrix}$$

where δ is the necessary adjustment of ω_c.

We seek the solution of (22) in the form:

(23) $$\begin{cases} I = I_o(T) + \mu[I_1(T) + U_{v1}(t,T)] + \mu^2[I_2(T) + U_{v2}(t,T)] + \dots \\ \theta = \theta_o(T) + \mu[\theta_1(T) + U_{w1}(t,T)] + \mu^2[\theta_2(T) + U_{w2}(t,T)] + \dots \\ \delta = \mu\delta_1 + \mu^2\delta_2 + \mu^3\delta_3 + \dots \\ \dfrac{d\theta_o(t)}{dt} = \omega_o(T); \quad \dfrac{d\theta_1(t)}{dt} = \omega_1(T); \quad \dfrac{d\theta_2(t)}{dt} = \omega_2(T); \dots \end{cases}$$

where $U_{Vk}\ (t,T)$ and $U_{Wk}\ (t,T)$ do not contain secular terms, i.e.

(24) $$\left\langle \frac{\partial}{\partial t} U_{vk}(t,T) \right\rangle_t = 0, \quad \left\langle \frac{\partial}{\partial t} U_{wk}(t,T) \right\rangle_t = 0, \quad k = 1, 2, 3, \dots$$

Substituting (23) in (22), developing in a series in the powers of μ and equating the coefficients multiplying the same powers of μ^k we get:

- in front of μ^o

$\dfrac{d\theta_o(t)}{dt} = \omega_{oo}(T)$, where $\omega_{oo}(T) = \omega_C(I_{oo}, T)$ the initial value I_{oo} is taken at the moment t = 0, i.e.

$$I_{oo} = I(T)|_{t=0}$$

or $$\theta_o(t) = \int_0^t \omega_c(I_{oo}, T + \mu t')dt' + const$$

- in front of μ^1

(25) $$\frac{dI_o(T)}{dT} + \frac{\partial U_{v1}(t,T)}{\partial t} = G_v(I_o(T), \theta_o(T), t, T, 0)$$

(26) $$\omega_1(T) + \frac{\partial U_{w1}(t,T)}{\partial t} = \delta_1 + G_w(I_o(T), \theta_o(T), t, T, 0)$$

and accordingly $$\delta_1 = \left[\frac{\omega_c(I_o(T), T - \omega_c(I_{oo}, T)}{\mu}\right]$$

$$\delta_2 = \left[\frac{\omega_c(I_o(T) + \mu I_1(T) + \mu U_{v1}(t,T)) - \omega_c(I_o(T), T)}{\mu^2}\right]$$

etc.

Averaging the two sides of (25) and (26) and taking into account (24) we obtain:

(27) $$\frac{dI_o(T)}{dT} = \langle G_v[I_o(T), \theta_o(T), t, T, 0] \rangle_t$$

$$\omega_1(T) = \left\langle G_w[I_{\cdot\cdot}(T), \theta_{\cdot\cdot}(T), t, T, 0] + \delta_1 \right\rangle_t$$

(28)

$$\theta_1(t) = \int_0^t \omega_1(T_o + \mu t')dt' + const$$

The differential equation (26) can be resolved, i.e. through successive approximations and development of T in a power series. The determined by (27) quantity $I_o(\Psi)$ is then substituted in (28).

So, we have come on the approximation as follows:

(29)

$$\begin{bmatrix} \left\langle \dfrac{dI}{dt} \right\rangle_t \\ \left\langle \dfrac{d\theta}{dt} \right\rangle_t \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_o \end{bmatrix} + \mu \left\langle \mathbf{Z}^{-1} \begin{bmatrix} -\dfrac{\partial x_c}{\partial T} \\ -\dfrac{\partial p_c}{\partial T} + \mu F_v \end{bmatrix} \right\rangle$$

where [$\omega_o = \omega_c$ [$I_o(T)$, T]= $\omega_{oo} + \mu\,\delta_1$

Then

$$U_{v1}(t,T) = \int_0^t \left[G_{v1} - \left\langle G_{v1} \right\rangle_t \right] dt' + const$$

$$U_{w1}(t,T) = \int_0^t \left[G_{w1} + \delta_1 - \left\langle G_{w1} + \delta_1 \right\rangle_t \right] dt' + const$$

4. Solution by Kuzmak's method in matrix form

In the two methods developed above we first transformed to functions with a constant period $p_o = 2\pi$ and only then introduced matrices **Y** and **Z**. This is why **Y** and **Z** turned out to be periodic. With Kuzmak's method [18] one proceeds in the reverse order: first, the functional square matrix Ξ is introduced and only then the transformation to constant period functions is performed. As a result, Ξ is not periodic but contains secular terms. In the analysis of the perturbed equation these terms compensate each other and the final solution is periodic. Here, we will develop a version of Kuzmak's method on the basis of a matrix presentation in application to an asynchronous non-linear oscillator.

The generating solution of the system of equations (1) when $\mu \equiv 0$ is assumed to be in form (2).

We introduce the matrix:

(30)

$$\Xi(E, t + t_o, T) = \begin{bmatrix} \dfrac{\partial x_o(E, t + t_o, T)}{\partial E} & \dfrac{\partial x_o(E, t + t_o, T)}{\partial (t + t_o)} \\ \dfrac{\partial p_o(E, t + t_o, T)}{\partial E} & \dfrac{\partial p_o(E, t + t_o, T)}{\partial (t + t_o)} \end{bmatrix},$$

so that the following variation equation is satisfied:

(31) $$\frac{\partial \Xi}{\partial t} + \mathbf{B}\,\Xi = 0, \text{ where } \mathbf{B} = \begin{pmatrix} 0 & -1 \\ \frac{\partial f}{\partial x_o} & 0 \end{pmatrix}.$$

Besides

(32) $$\Xi = \mathbf{Y}\,\mathbf{H}\,[1 - (t+t_o)\,\mathbf{Q}\,],$$

where $$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 0 & 0 \\ \frac{\partial \ln \Pi(E,T)}{\partial E} & 0 \end{bmatrix}$$

Then det $\Xi = -1$ and $\Xi^{-1}(E, t+t_o, T) = [1 + (t+t_o)\mathbf{Q}]\,\mathbf{H}^{-1}\mathbf{Y}^{-1}$.

The solution of the perturbed system of equations (1) we seek in the form:

$$\begin{cases} x = x_b\ [E(T),\ \Psi(t) + \alpha(T), T\,] + \mu\, U_{1a}\ (t,T) \\ p = p_b\ [E(T),\ \Psi(t) + \alpha(T), T\,] + \mu\, U_{2a}\ (t,T) \end{cases}$$

where the constant parameters Ψ, E, α, θ, and dΨ/dt = ω(E,T) are varied. We lay down conditions U_{1a} and U_{2a} not to contain secular terms.

The following equation is satisfied:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dp}{dt} \end{bmatrix} = \begin{bmatrix} \omega \frac{\partial x_b}{d\theta} \\ \omega \frac{dp_b}{d\theta} \end{bmatrix} + \mu \mathbf{Y} \begin{bmatrix} \frac{dE}{dT} \\ \frac{d\alpha}{dT} \end{bmatrix} + \mu \begin{bmatrix} \frac{\partial x_b}{dT} + \frac{\partial U_{1a}}{\partial t} + \mu \frac{\partial U_{1a}}{\partial T} \\ \frac{\partial p_b}{dT} + \frac{\partial U_{2a}}{\partial t} + \mu \frac{\partial U_{2a}}{\partial T} \end{bmatrix}.$$

We also introduce the matrix:

$$\mathrm{U_a}(t,T) = \begin{bmatrix} \mathrm{U_{a1}}\ (t,T) \\ \mathrm{U_{a2}}(t,T) \end{bmatrix}$$

We seek the solution in the form of an asymptotic series:

(33) $$\begin{cases} \Psi(t) = \Psi_o(t) + \mu \Psi_1(t) + \mu^2 \Psi_2(t) + \ldots \\ \alpha(T) = \alpha_o(T) + \mu\alpha_1(T) + \mu^2 \alpha_2(T) + \ldots \\ \theta(t,T) = \theta_o(t,T) + \mu\theta_1(t,T) + \mu^2 \theta_2(t,T) + \ldots \\ \theta_k(t,T) = \varphi_k(t) + \alpha_k(t), \qquad k = 0, 1, 2, \ldots \\ E(T) = E_o(T) + \mu E_1(T) + \mu^2 E_2(T) + \ldots \\ \mu U_a(t,T) = \mu U_1(t,T) + \mu^2 U_2(t,T) + \ldots \\ U_k(t,T) = \begin{bmatrix} U_{k1}(t,T) \\ U_{k2}(t,T) \end{bmatrix}, \qquad k = 1, 2, 3, \ldots \end{cases}$$

We set as a goal that U_{k1}, U_{k2} must not contain secular terms.

The generating solution has the form (5).

Substituting (33) in (1) and taking (5) into account developing in a series in the powers of μ and equating the coefficients multiplying the same powers of μ^k, we get:

- in front of μ^o

(34) $$\frac{d\Psi_o(t)}{dt} = \omega(E_{oo},T)$$

$$d\Psi_o = \int_0^t \omega(E_{oo},T_o+\mu t')dt'$$

where E_{oo} is the initial value of $E_o\ (T)$ at the moment $t = 0$ and when $T = T_o$;

- in front of μ^k

(35) $$\frac{\partial \mathbf{U}_k(t,T)}{\partial t} + \mathbf{B}\mathbf{U}_k(t,T) = \mathbf{\Phi}_k; \qquad \kappa = 1,2,3,\ldots$$

where $$\mathbf{\Phi}_k = col(\Phi_{k1},\Phi_{k2},)\frac{d\Psi_k(t)}{dt} = \delta_k(T_o + \mu t), \qquad k = 1,2,3,\ldots$$

(36) $$\Psi_k(t) = \int_0^t \delta_k(T+\mu t')dt'$$

δ_k are the necessary adjustments of ω

$$\delta_1(T_o+\mu t) = \delta_1(T) = \frac{\omega[E_o(T),T] - \omega[E_{oo},T]}{\mu}$$

$$\delta_k(T_o+\mu t) = \delta_k(T) = \frac{\omega\left[E_o(T)+\mu E_1(T)+\ldots+\mu^{k-1}E_{k-1}(T)\right] - \omega\left[E_o(T)+\mu E_1(T)+\ldots+\mu^{k-2}E_{k-2}(T)\right]}{\mu^k}$$

, k=2,3,4,.....

In particular, for the coefficient multiplying μ^1 we obtain:

$$\frac{\partial \mathrm{U}_1(t,T)}{\partial t} + \mathrm{B}\mathrm{U}_1(t,T) = \Phi_1;$$

where

(37) $$\mathbf{\Phi}_1 = -\mathbf{Y}\begin{bmatrix} \dfrac{dE_o(T)}{dT} \\ \dfrac{d\alpha(T)}{dT} \end{bmatrix} + \begin{bmatrix} -\dfrac{\partial x_b}{\partial T} \\ -\dfrac{\partial p_b}{\partial T} + F_v \end{bmatrix}$$

Here $F = F(\frac{dx_b}{dt}, x_b, t, T, 0)$.

We seek the solution of equation (35) in the form:

(38) $$\mathbf{U}_k(t,T) = \Xi \mathbf{V}_k(t,T), \qquad k = 1,2,3,...$$

where

$$\mathbf{V}_k(t,T) = \begin{bmatrix} V_{k1}(t,T) \\ V_{k2}(t,T) \end{bmatrix}, \qquad k = 1,2,3,...$$

Substituting (38) into (35) and taking (31) into account we get:

(39) $$\mathbf{V}_k(t,T) = \mathbf{V}_k(0,T) + \int_0^t \Xi^{-1}(E, t'+t'', T)\Phi(t',T)dt'$$

From equation (14) it follows:

(40) $$\mathbf{V}(t,T) = \mathbf{V}(0,T) + \int_0^t [1 + (t'+t_o)\mathbf{Q}]\mathbf{H}^{-1}\mathbf{Y}^{-1}\Phi dt'$$

$$= \mathbf{V}(0,T) + \int_0^t [1 + (t'+t_o)\mathbf{Q}]\{\frac{\partial}{\partial t'}[\mathrm{K}_1 + \mathbf{D}(T)t']\}dt'$$

where matrices $\mathbf{D}$(T) and $\mathbf{K}_1$ have been introduced through the relations:

(41) $$\mathrm{D}(T) = \langle \mathrm{H}^{-1}\mathrm{Y}^{-1}\Phi \rangle_t$$

$$\int_0^t \mathrm{H}^{-1}\mathrm{Y}^{-1}\Phi dt' = \mathrm{K}_1[\Psi(t), T] + \mathrm{D}(T)t$$

for $\mathbf{K}_1$ $[\Psi(0),\mathrm{T}] = 0$.

Integrating (40) by parts we get:

$$\mathrm{V}(t,T) = \mathrm{V}(0,T) + [-1 + (t + t_o)\mathrm{Q}][\mathrm{K}_1 + \mathrm{D}(t)t] - \mathrm{Q}[\mathrm{K}_2 + \mathrm{L}(t)t + \frac{Dt^2}{2}]$$

where $\mathbf{L}(t) = \langle \mathbf{K}_1 \rangle_t$ and

$$\int_0^t \mathrm{K}_1[\Psi(t'), T]dt' = \mathrm{K}_2[\Psi(t), T] + \mathrm{L}(T)t$$

Substituting in (38) and taking into account (32) as well as the fact that $\mathbf{Q}^2=0$, we obtain:

$$U(t,T) = YH\{\frac{QDt^2}{2} + [D - QL(T) - QV(0,T)]t + K_1 - QK_2 + (1 - t_0Q)V(0,T)\}$$

The matrix function $\boldsymbol{U}$ will be periodic in t if $\boldsymbol{QD}$=0 and $\boldsymbol{D}$(T) = $\boldsymbol{Q}$ [$\boldsymbol{L}$(T) + (0,T)] to the satisfaction of which it is sufficient to do the substitution:

(42), (43) $\boldsymbol{D}=0, \quad \boldsymbol{V}(0,T) = -\boldsymbol{L}(T)$

In doing this we obtain:

(44) $\boldsymbol{U} = \boldsymbol{YH}\ [\boldsymbol{K_1} - \boldsymbol{QK_2} + (1 - t_0\ \boldsymbol{Q})\ \boldsymbol{V}(0,T)]$

Taking into account definition (41), condition (44) is equivalent to:

$<Y^{-1}\Phi_k>_t = 0 \qquad k=1,2,3, \ldots$

In particular, when k = 1 from (37) and (45) it follows:

(46)
$$\begin{bmatrix} \frac{dE_o(T)}{dT} \\ \frac{d\alpha_o(T)}{dT} \end{bmatrix} = \left\langle Y^{-1} \begin{bmatrix} -\frac{\partial x_b}{\partial T} \\ -\frac{\partial p_b}{\partial T} + F_v \end{bmatrix} \right\rangle_t$$

In this way we obtained a system of equations (36), (43) and (46) which in addition serve as a basis of comparison with the results obtained with the non-canonical perturbation approach in energy-angle variables.

Conclusion

We conclude with the important observation that to first approximation the solution in energy-angle variables coincides with the solution to first order in action-angle variables as well as with the solution obtained by Kuzmak's method.

Indeed if in (29) we substitute $\omega = \omega_c = \frac{\partial E(I,T)}{\partial I}$ for $I = I_o\ (T)$, i.e. $\omega = \omega_o$ as well as $Z = Y\begin{pmatrix} \omega_c & 0 \\ 0 & 1 \end{pmatrix}$, we obtain:

(47)
$$\begin{bmatrix} \omega\left\langle \frac{dI}{dt} \right\rangle \\ \omega\left\langle \frac{d\theta}{dt} \right\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_o \end{bmatrix} + \mu\left\langle Y^{-1} \begin{bmatrix} -\frac{\partial x_c}{\partial T} \\ -\frac{\partial p_c}{\partial T} + F_v \end{bmatrix} \right\rangle_t$$

The comparison of equation (47) with the averaged equation (8) confirms the above conclusion. An analogous conclusion can be obtained through analysis of and comparison with equation (46).

The obtained results contribute in support of the idea that, in particular in the analysis of an oscillator under external asynchronous influence there isn't any significant difference between the non-canonical (non-Hamiltonian) and the canonical (Hamiltonian) methods. It is necessary, though, to mention that a number of other methods exist in the theory of non-linear oscillations which are not even to first order completely equivalent with the solution obtained by the considered above three methods.

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НЕЛИНЕЕН ОСЦИЛАТОР ПОД ВЪНШНО АСИНХРОННО ВЪЗДЕЙСТВИЕ: СРАВНЕНИЕ НА КАНОНИЧНИТЕ И НЕКАНОНИЧНИТЕ ПЕРТУРБАЦИОННИ МЕТОДИ ЗА АНАЛИЗ

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Резюме

В статията е представен неканоничен (нехамилтоновски) пертурбационен метод за изследване на нелинеен осцилатор под външно асинхронно въздействие с променливи "енергия-ъгъл". Като нови променливи са въведени итерационните константи на първоначалното решение. Прилагайки последователно метода на каноничните трансформации и получавайки функциите, е разработен каноничен метод с променливи "действие-ъгъл" за анализ на същата система в подобни условия. Двата метода се характеризират с извършване още в началото на преход към функции с постоянен период, като едва след това се въвеждат необходимите матрици на функционала. Същият проблем е изследван по метода на Кузмак, който се отличава с обратния подход - най-напред се въвежда квадратната матрица на функционала, и едва след това се осъществява прехода към функции с постоянен период. Направено е сравнение на резултатите, получени при използване на трите гореспоменати метода. Показано е, че решенията в първо приближение водят до еднакви резултати. Конкретно, това заключение е принос към идеята, че няма съществена разлика между неканоничните (нехамилтоновски) и каноничните (хамилтоновски) методи. Обръща се внимание, обаче, на факта, че другите методи, разработени в рамките на теорията на нелинейните колебания, не могат да дадат дори в първо приближение пълно съвпадение с решенията, получени при използването на трите гореспоменати метода.